14.6 | Bernoulli's Equation

Learning Objectives

By the end of this section, you will be able to:

- Explain the terms in Bernoulli's equation
- Explain how Bernoulli's equation is related to the conservation of energy
- Describe how to derive Bernoulli's principle from Bernoulli's equation
- Perform calculations using Bernoulli's principle
- Describe some applications of Bernoulli's principle

As we showed in **Figure 14.27**, when a fluid flows into a narrower channel, its speed increases. That means its kinetic energy also increases. The increased kinetic energy comes from the net work done on the fluid to push it into the channel. Also, if the fluid changes vertical position, work is done on the fluid by the gravitational force.

A pressure difference occurs when the channel narrows. This pressure difference results in a net force on the fluid because the pressure times the area equals the force, and this net force does work. Recall the work-energy theorem,

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

The net work done increases the fluid's kinetic energy. As a result, the pressure drops in a rapidly moving fluid whether or not the fluid is confined to a tube.

There are many common examples of pressure dropping in rapidly moving fluids. For instance, shower curtains have a disagreeable habit of bulging into the shower stall when the shower is on. The reason is that the high-velocity stream of water and air creates a region of lower pressure inside the shower, whereas the pressure on the other side remains at the standard atmospheric pressure. This pressure difference results in a net force, pushing the curtain inward. Similarly, when a car passes a truck on the highway, the two vehicles seem to pull toward each other. The reason is the same: The high velocity of the air between the car and the truck creates a region of lower pressure between the vehicles, and they are pushed together by greater pressure on the outside (**Figure 14.29**). This effect was observed as far back as the mid-1800s, when it was found that trains passing in opposite directions tipped precariously toward one another.

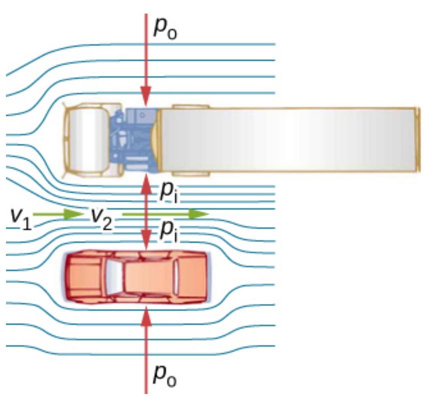

Figure 14.29 An overhead view of a car passing a truck on a highway. Air passing between the vehicles flows in a narrower channel and must increase its speed (v_2 is greater than v_1), causing the pressure between them to drop (p_i is less than p_o). Greater pressure on the outside pushes the car and truck together.

Energy Conservation and Bernoulli's Equation

The application of the principle of conservation of energy to frictionless laminar flow leads to a very useful relation between pressure and flow speed in a fluid. This relation is called **Bernoulli's equation**, named after Daniel Bernoulli (1700–1782), who published his studies on fluid motion in his book *Hydrodynamica* (1738).

Consider an incompressible fluid flowing through a pipe that has a varying diameter and height, as shown in **Figure 14.30**. Subscripts 1 and 2 in the figure denote two locations along the pipe and illustrate the relationships between the areas of the cross sections A, the speed of flow v, the height from ground y, and the pressure p at each point. We assume here that the density at the two points is the same—therefore, density is denoted by ρ without any subscripts—and since the fluid in incompressible, the shaded volumes must be equal.

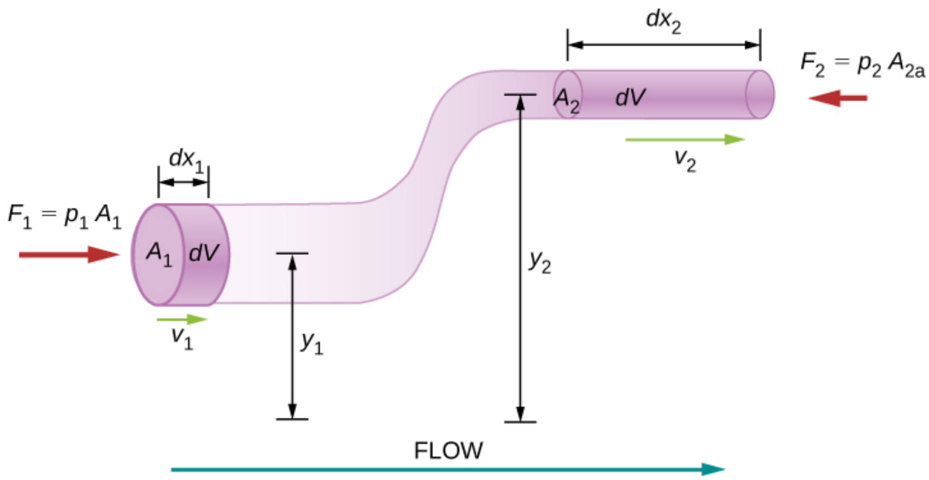

Figure 14.30 The geometry used for the derivation of Bernoulli's equation.

We also assume that there are no viscous forces in the fluid, so the energy of any part of the fluid will be conserved. To derive Bernoulli's equation, we first calculate the work that was done on the fluid:

$$dW = F_1 dx_1 - F_2 dx_2$$
$$dW = p_1 A_1 dx_1 - p_2 A_2 dx_2 = p_1 dV - p_2 dV = (p_1 - p_2)dV.$$

The work done was due to the conservative force of gravity and the change in the kinetic energy of the fluid. The change in the kinetic energy of the fluid is equal to

$$dK = \frac{1}{2}m_2 v_2^2 - \frac{1}{2}m_1 v_1^2 = \frac{1}{2}\rho dV\left(v_2^2 - v_1^2\right).$$

The change in potential energy is

$$dU = mgy_2 - mgy_1 = \rho dVg(y_2 - y_1).$$

The energy equation then becomes

$$\begin{aligned} dW &= dK + dU \\ (p_1 - p_2)dV &= \frac{1}{2}\rho dV\left(v_2^2 - v_1^2\right) + \rho dVg(y_2 - y_1) \\ (p_1 - p_2) &= \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) + \rho g(y_2 - y_1). \end{aligned}$$

Rearranging the equation gives Bernoulli's equation:

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2.$$

This relation states that the mechanical energy of any part of the fluid changes as a result of the work done by the fluid external to that part, due to varying pressure along the way. Since the two points were chosen arbitrarily, we can write Bernoulli's equation more generally as a conservation principle along the flow.

Bernoulli's Equation

For an incompressible, frictionless fluid, the combination of pressure and the sum of kinetic and potential energy densities is constant not only over time, but also along a streamline:

$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{constant} \tag{14.16}$$

A special note must be made here of the fact that in a dynamic situation, the pressures at the same height in different parts of the fluid may be different if they have different speeds of flow.

Analyzing Bernoulli's Equation

According to Bernoulli's equation, if we follow a small volume of fluid along its path, various quantities in the sum may change, but the total remains constant. Bernoulli's equation is, in fact, just a convenient statement of conservation of energy for an incompressible fluid in the absence of friction.

The general form of Bernoulli's equation has three terms in it, and it is broadly applicable. To understand it better, let us consider some specific situations that simplify and illustrate its use and meaning.

Bernoulli's equation for static fluids

First consider the very simple situation where the fluid is static—that is, $v_1 = v_2 = 0.$ Bernoulli's equation in that case is

$$p_1 + \rho g h_1 = p_2 + \rho g h_2.$$

We can further simplify the equation by setting $h_2 = 0.$ (Any height can be chosen for a reference height of zero, as is often done for other situations involving gravitational force, making all other heights relative.) In this case, we get

$$p_2 = p_1 + \rho g h_1.$$

This equation tells us that, in static fluids, pressure increases with depth. As we go from point 1 to point 2 in the fluid, the depth increases by h_1 , and consequently, p_2 is greater than p_1 by an amount $\rho g h_1$. In the very simplest case, p_1 is zero at the top of the fluid, and we get the familiar relationship $p = \rho g h$. (Recall that $p = \rho g h$ and $\Delta U_\text{g} = -mgh.$) Thus, Bernoulli's equation confirms the fact that the pressure change due to the weight of a fluid is $\rho g h$. Although we introduce Bernoulli's equation for fluid motion, it includes much of what we studied for static fluids earlier.

Bernoulli's principle

Suppose a fluid is moving but its depth is constant—that is, $h_1 = h_2$. Under this condition, Bernoulli's equation becomes

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2.$$

Situations in which fluid flows at a constant depth are so common that this equation is often also called **Bernoulli's principle**, which is simply Bernoulli's equation for fluids at constant depth. (Note again that this applies to a small volume of fluid as we follow it along its path.) Bernoulli's principle reinforces the fact that pressure drops as speed increases in a moving fluid: If v_2 is greater than v_1 in the equation, then p_2 must be less than p_1 for the equality to hold.

Example 14.6

Calculating Pressure

In Example 14.5, we found that the speed of water in a hose increased from 1.96 m/s to 25.5 m/s going from the hose to the nozzle. Calculate the pressure in the hose, given that the absolute pressure in the nozzle is $1.01 \times 10^5 \text{ N/m}^2$ (atmospheric, as it must be) and assuming level, frictionless flow.

Strategy

Level flow means constant depth, so Bernoulli's principle applies. We use the subscript 1 for values in the hose and 2 for those in the nozzle. We are thus asked to find $p1$.

Solution

Solving Bernoulli's principle for p_1 yields

$$p_1 = p_2 + \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho(v_2^2 - v_1^2).$$

Substituting known values,

$$\begin{aligned} p_1 &= 1.01 \times 10^5 \text{ N/m}^2 + \tfrac{1}{2}(10^3 \text{ kg/m}^3)[(25.5 \text{ m/s})^2 - (1.96 \text{ m/s})^2] \\ &= 4.24 \times 10^5 \text{ N/m}^2. \end{aligned}$$

Significance

This absolute pressure in the hose is greater than in the nozzle, as expected, since v is greater in the nozzle. The pressure p_2 in the nozzle must be atmospheric, because the water emerges into the atmosphere without other changes in conditions.

Applications of Bernoulli's Principle

Many devices and situations occur in which fluid flows at a constant height and thus can be analyzed with Bernoulli's principle.

Entrainment

People have long put the Bernoulli principle to work by using reduced pressure in high-velocity fluids to move things about. With a higher pressure on the outside, the high-velocity fluid forces other fluids into the stream. This process is called *entrainment*. Entrainment devices have been in use since ancient times as pumps to raise water to small heights, as is necessary for draining swamps, fields, or other low-lying areas. Some other devices that use the concept of entrainment are shown in **Figure 14.31**.

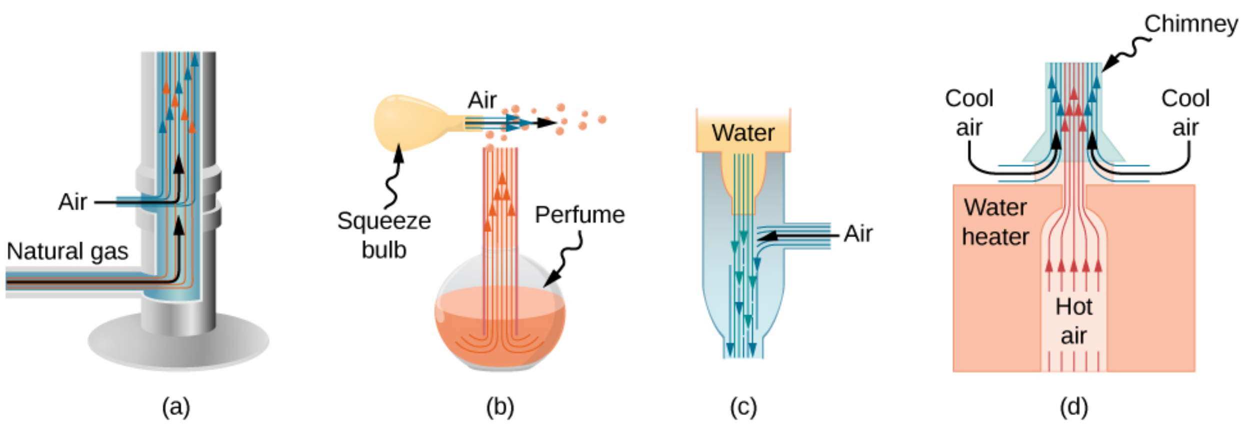

Figure 14.31 Entrainment devices use increased fluid speed to create low pressures, which then entrain one fluid into another. (a) A Bunsen burner uses an adjustable gas nozzle, entraining air for proper combustion. (b) An atomizer uses a squeeze bulb to create a jet of air that entrains drops of perfume. Paint sprayers and carburetors use very similar techniques to move their respective liquids. (c) A common aspirator uses a high-speed stream of water to create a region of lower pressure. Aspirators may be used as suction pumps in dental and surgical situations or for draining a flooded basement or producing a reduced pressure in a vessel. (d) The chimney of a water heater is designed to entrain air into the pipe leading through the ceiling.

Velocity measurement

Figure 14.32 shows two devices that apply Bernoulli's principle to measure fluid velocity. The manometer in part (a) is connected to two tubes that are small enough not to appreciably disturb the flow. The tube facing the oncoming fluid creates a dead spot having zero velocity ($v_1 = 0$) in front of it, while fluid passing the other tube has velocity v_2 . This means that Bernoulli's principle as stated in

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

becomes

$$p_1 = p_2 + \frac{1}{2}\rho v_2^2.$$

Thus pressure p_2 over the second opening is reduced by $\frac{1}{2}\rho v_2^2$, so the fluid in the manometer rises by h on the side connected to the second opening, where

$$h \propto \frac{1}{2}\rho v_2^2.$$

(Recall that the symbol $\propto$ means "proportional to.") Solving for v_2, we see that

$$v_2 \propto \sqrt{h}.$$

Part (b) shows a version of this device that is in common use for measuring various fluid velocities; such devices are frequently used as air-speed indicators in aircraft.

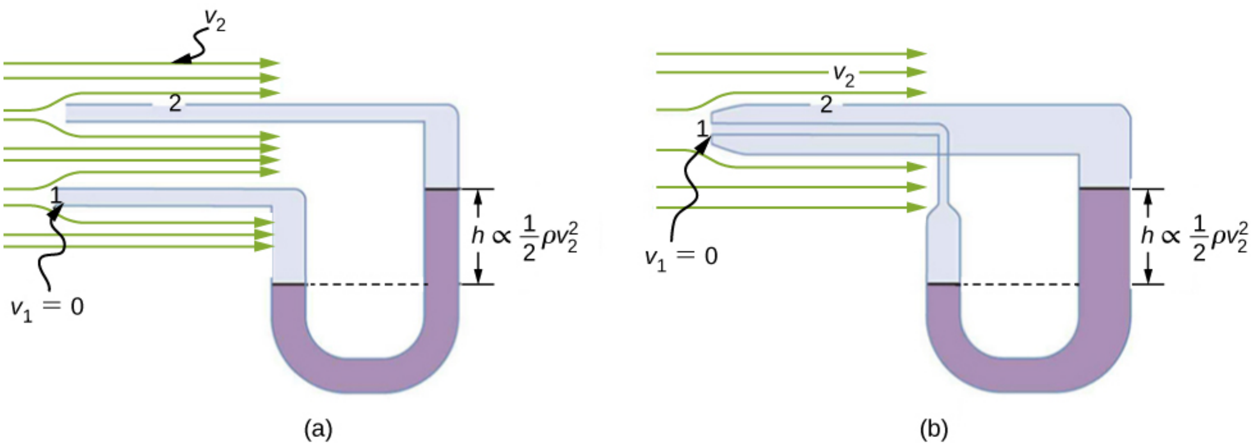

Figure 14.32 Measurement of fluid speed based on Bernoulli's principle. (a) A manometer is connected to two tubes that are close together and small enough not to disturb the flow. Tube 1 is open at the end facing the flow. A dead spot having zero speed is created there. Tube 2 has an opening on the side, so the fluid has a speed v across the opening; thus, pressure there drops. The difference in pressure at the manometer is $\frac{1}{2}\rho v_2^2$, so h is proportional to $\frac{1}{2}\rho v_2^2$. (b) This type of velocity measuring device is a Prandtl tube, also known as a pitot tube.

A fire hose

All preceding applications of Bernoulli's equation involved simplifying conditions, such as constant height or constant pressure. The next example is a more general application of Bernoulli's equation in which pressure, velocity, and height all change.

Example 14.7

Calculating Pressure: A Fire Hose Nozzle

Fire hoses used in major structural fires have an inside diameter of 6.40 cm (**Figure 14.33**). Suppose such a hose carries a flow of 40.0 L/s, starting at a gauge pressure of $1.62 \times 10^6 \text{ N/m}^2$. The hose rises up 10.0 m along a ladder to a nozzle having an inside diameter of 3.00 cm. What is the pressure in the nozzle?

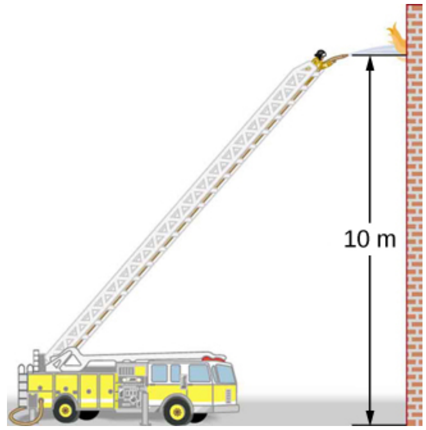

Figure 14.33 Pressure in the nozzle of this fire hose is less than at ground level for two reasons: The water has to go uphill to get to the nozzle, and speed increases in the nozzle. In spite of its lowered pressure, the water can exert a large force on anything it strikes by virtue of its kinetic energy. Pressure in the water stream becomes equal to atmospheric pressure once it emerges into the air.

Strategy

We must use Bernoulli's equation to solve for the pressure, since depth is not constant.

Solution

Bernoulli's equation is

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

where subscripts 1 and 2 refer to the initial conditions at ground level and the final conditions inside the nozzle, respectively. We must first find the speeds v_1 and v_2. Since $Q = A_1 v_1$, we get

$$v_1 = \frac{Q}{A_1} = \frac{40.0 \times 10^{-3}\,\text{m}^3/\text{s}}{\pi(3.20 \times 10^{-2}\,\text{m})^2} = 12.4\text{m/s}.$$

Similarly, we find

$$v_2 = 56.6\text{ m/s}.$$

This rather large speed is helpful in reaching the fire. Now, taking h_1 to be zero, we solve Bernoulli's equation for p_2:

$$p_2 = p_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) - \rho g h_2.$$

Substituting known values yields

$$\begin{aligned} p_2 &= 1.62 \times 10^6\text{ N/m}^2 + \frac{1}{2}(1000\text{ kg/m}^3)[(12.4\text{ m/s})^2 - (56.6\text{ m/s})^2] \\ &- (1000\text{ kg/m}^3)(9.80\text{ m/s}^2)(10.0\text{ m}) \\ &= 0. \end{aligned}$$

Significance

This value is a gauge pressure, since the initial pressure was given as a gauge pressure. Thus, the nozzle pressure equals atmospheric pressure as it must, because the water exits into the atmosphere without changes in its conditions.